GAGA

Jujian Zhang

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Chapter 1

Commutative Algebra

In this chapter, we collect some results of commutative algebra.

Theorem 1.1 (Hilbert's Nullstellensatz)**.** *Let k be a field and A be a finitely generated k-algebra. Then for every prime ideal $\mathfrak{p}$ of A,*

$$\mathfrak{p} = \bigcap_{\mathfrak{p} \leq \mathfrak{m} \text{ is maximal}} \mathfrak{m}.$$

If $\mathfrak{m}$ is a maximal ideal of A, then $k \hookrightarrow A \big/ \mathfrak{m}$ is a finite field extension.

Proof. □

Corollary 1.2. *Let k be an algebraically closed filed and A a finitely generated k-algebra, then $A \big/ \mathfrak{m}$ is isomorphic to k.*

Proof. The field extension $A \big/ \mathfrak{m}$ over k is finite hence algebraic. Since k is algebraically closed, $A \big/ \mathfrak{m}$ is isomorphic to k. □

Corollary 1.3. *Let k be an algebraically closed field and A a finitely generated k-algebras, then each maximal ideals $\mathfrak{m}$ A is the kernel of a unique algebra homomorphism $\phi_{\mathfrak{m}} : A \to k$*

Proof. Existence: the kernel of $A \xrightarrow{\pi} A \big/ \mathfrak{m} \xrightarrow{\sim} k$ is exactly $\mathfrak{m}$.

Uniqueness: assume ϕ and ψ are two $\mathbb{C}$-algebra homomorphism such that $\mathfrak{m} = \ker \phi = \ker \psi$. Let $\rho : \mathbb{C} \to R$ be the structure map of R. For any $r \in R$, we claim that there exists some $x \in \mathfrak{m}$ and $\lambda \in \mathbb{C}$, such that $r = x + \rho(\lambda)$. Indeed, we have that $r = (r - \rho(\phi(r))) + \rho(\phi(r))$, and $r - \rho(\phi(r))$ is in the kernel of ϕ. Thus for every $r = m + \rho(\lambda)$, we have $\phi(r) = \phi(\rho(\lambda)) = \psi(\rho(\lambda)) = \psi(r)$. □

Chapter 2

Schemes of finite type over $\mathbb{C}$

2.1 Some general scheme theory

Before we can define our main object of interest, *schemes of finite type over* $\mathbb{C}$, we need to introduce some preliminary notions.

We already have the following in mathlib4 thanks to Andrew Yang:

Definition 2.1 (Local property of ring homomorphisms)**.** *Let P be a property of ring homomorphisms: we say the property P is local if*

1. *if P holds for $\phi : A \to B$, then P holds for $\phi_S : S^{-1}A \to \langle f(S) \rangle^{-1} B$ for any submonoid $S \subseteq A$.*
2. *Let $\phi : A \to B$ be a ring homomorphism, if P holds for $A \xrightarrow{\phi} B \to B_{f_i}$ for some $\{f_i\} \subseteq B$ such that $\langle f_i \rangle = B$, then P holds for ϕ.*
3. *Let $\phi : A \to B$ and $\psi : B \to C$ be two ring homomorphisms, if P holds for ϕ and ψ, then P holds for $\psi \circ \phi$.*
4. *P holds for $A \to A_f$ for all $f \in A$.*

Proposition 2.2. *The property "finite type" of ring homomorphisms is local in the sense of Definition 2.1.*

Definition 2.3. *If P is a property of ring homomorphisms then the property* affine locally P *of scheme morphism $(\phi, \phi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ holds if and only if P holds for all ring homomorphism $\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y)$ for all affine subsets $U \subseteq X$ and $V \subseteq Y$ such that $\phi(U) \leq V$.*

Definition 2.4 (Morphisms locally of finite type [3, 01T0])**.** *Let $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) := (\phi, \phi^*)$ be a morphism of schemes. We say*

1. *Φ is locally of finite type if for any affine open $V \subseteq Y$ and affine open $U \subseteq X$ such that $\phi(U) \subseteq V$, we have the induced map $\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_V)$ is a ring map of finite type. In another word, Φ is affine locally a ring homomorphism of finite type.*
2. *Φ is of finite type if it is locally of finite type and ϕ is quasi-compact.*

Proposition 2.5. *Let* $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be an open immersion between schemes, then* Φ *is locally of finite type.*

Proposition 2.6. *Let* $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be a morphism of schemes and* $\mathcal{U} = \{U_i\}$ *be an open covering of* X*, then affine locally* Φ *is locally of finite type if and only if* $\Phi|_{U_i}$ *is a morphism locally of finite type.*

Proposition 2.7. *Let* $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be a morphism of schemes and* $\mathcal{U} = \{U_i\}$ *be an affine open covering of* Y*. Consider the pullback cover* $\mathcal{V} = \{V_i\}$ *of* X*, and if for each* i*, there is an affine open cover* $\mathcal{W}_i = \{W_{i,j}\}$ *for* $V_i \subseteq X$*. Then* Φ *is locally of finite type if and only if, for each* i *and* j*, the ring map* $\Gamma(W_{i,j}, \mathcal{O}_X) \to \Gamma(U_i, \mathcal{O}_Y)$ *is of finite type.*

Proposition 2.8. *Composition of morphisms locally of finite type is locally of finite type.*

2.2 Basic definitions and properties

Definition 2.9 (Schemes locally of finite type over $\mathbb{C}$)**.** *A scheme* $(X, \mathcal{O}_X)$ *is locally of finite type over* $\mathbb{C}$ *if* $(X, \mathcal{O}_X)$ *is a scheme over* $\mathbb{C}$ *and the structure morphism* $(X, \mathcal{O}_X) \to \left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right)$ *is a morphism locally of finite type.*

Definition 2.10 (Schemes of finite type over $\mathbb{C}$)**.** *A scheme* $(X, \mathcal{O}_X)$ *is of finite type over* $\mathbb{C}$ *if* $(X, \mathcal{O}_X)$ *is locally of finite type over* $\mathbb{C}$ *and the structure morphism is quasicompact.*

Let us unpack the Definition 2.9 a little:

Definition 2.11 (Affine open covering of spectra of finitely generated $\mathbb{C}$-algebras)**.** *An affine open covering of spectra of finitely generated* $\mathbb{C}$*-algebra for a scheme* $(X, \mathcal{O}_X)$ *over* $\mathbb{C}$ *is the following data:*

1. *indexing set:* I*;*
2. *a family of finitely generated algebras:* $R : I \to \mathsf{FGCAlg}_{\mathbb{C}}$*;*
3. *a family of open immersions: for each* $i \in I$*,* $\iota_i : \left(\operatorname{Spec} R_i, \widetilde{R_i}\right) \to (X, \mathcal{O}_X)$*;*
4. *covering:* $c : X \to I$ *such that for each* $x \in X$*,* $c_x \in \operatorname{range}(\iota_i)$*.*

Lemma 2.12. *A scheme* $(X, \mathcal{O}_X)$ *is locally of finite type over* $\mathbb{C}$ *if it is a scheme over* $\mathbb{C}$ *and it admits an affine open cover of spectra of finitely generated* $\mathbb{C}$*-algebras.*

Proof. This is unpacking Definition 2.9 □

Proposition 2.13. *Let* $(X, \mathcal{O}_X)$ *be a scheme locally of finite type over* $\mathbb{C}$*, let* $U \subseteq X$ *be an open subset, then* $(U, \mathcal{O}_X|_U)$ *is a scheme locally of finite type over* $\mathbb{C}$*.*

Proof. By Proposition 2.5 and Proposition 2.8, open immersions are locally of finite type and composition of morphisms locally of finite type is again locally of finite type, so

$$(U, \mathcal{O}_X|_U) \hookrightarrow (X, \mathcal{O}_X) \to \left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right)$$

is a morphism locally of finite type as well. □

Proposition 2.14. *If* $(X, \mathcal{O}_X)$ *is a scheme locally of finite type over* $\mathbb{C}$*, and* $U = \left(\operatorname{Spec} A, \widetilde{A}\right)$ *is an open affine subscheme of* $(X, \mathcal{O}_X)$*, then* $A \cong \Gamma(U, \mathcal{O}_X)$ *is a finitely generated* $\mathbb{C}$*-algebra as well.*

Proof. Consider the only open affine cover $\{\operatorname{Spec}\mathbb{C}\}$ of $\operatorname{Spec}\mathbb{C}$, its pullback cover is $\{X_{\mathbb{C}} := X \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}\}$ where $X_{\mathbb{C}}$ can be covered by $U_i \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}$ where U_i runs over the collection of all affine open sets. Then the conclusion follows from Proposition 2.7. □

Lemma 2.15. *If $U \subseteq \operatorname{Spec} A$ is an open subset where A is a finite $\mathbb{C}$-algebra, then U admits a finite covering of $D(f_1), \dots, D(f_n)$ where $D(x)$ is the basic open around $x \in A$.*

Proof. Since U is open, its complement U^{C} is of the form $V(I)$ for some ideal I. Since A is a finite $\mathbb{C}$-algebra, I is the span of $\{f_1, \dots, f_n\}$ for some $f_i \in A$. Thus $V(I)$ is $\bigcap_i V(f_i \cdot R)$ hence U is $\bigcup_i D(f_i)$. □

Lemma 2.16. *If $(X, \mathcal{O}_X)$ is a scheme of finite type over $\mathbb{C}$ and $V \subseteq X$ is an open subset, then V is quasi-compact.*

Proof. Since $(X, \mathcal{O}_X)$ is of finite type, it has a *finite* affine covering $\mathcal{U}_i = \{U_1, \dots, U_n\}$. It is sufficient to show that every open cover of $U_i \cap V$ has a finite subcover[1]. In another word, we only need to show if $(X, \mathcal{O}_X) \cong \left(\operatorname{Spec} A, \widetilde{A}\right)$ is affine and V is an open subset of X, then V is quasicompact. Since V is a *finite* union of $D(f_1), \dots, D(f_n)$ for some f_i's in A, we only need to show that $D(f)$ is quasicompact. Since $D(f)$ is affine, it is quasicompact. □

Corollary 2.17 (restriction of scheme of finite type over $\mathbb{C}$)**.** *Let $(X, \mathcal{O}_X)$ be a scheme of finite type over $\mathbb{C}$ and $U \subseteq X$ be open, then the restriction $(U, \mathcal{O}_X|_U)$ is a scheme of finite type over $\mathbb{C}$ as well.*

Proposition 2.18. *Let $\left(\operatorname{Spec} A, \widetilde{A}\right)$ and $\left(\operatorname{Spec} B, \widetilde{B}\right)$ be two affine finite schemes over $\mathbb{C}$. Then any morphism $\Phi : \left(\operatorname{Spec} A, \widetilde{A}\right) \to \left(\operatorname{Spec} B, \widetilde{B}\right)$ is of the form $(\phi, \phi^\star)$ where $\phi : B \to A$ is a $\mathbb{C}$-algebra homomorphism.*

Proof. We have a ring homomorphism ϕ, just need to check it commutes with algebra map. □

Maybe we should construct the following and recover the previous lemma as a corollary:

Proposition 2.19. *The category of affine scheme over $\mathbb{C}$ is antiquivalent to commutative $\mathbb{C}$-algebras. The category of affine finite scheme over $\mathbb{C}$ is antiequivalent to finitely generated commutative $\mathbb{C}$-algebras.*

$$\mathsf{ASch}_{\mathbb{C}}^{\mathsf{op}} \cong \mathsf{CAlg}_{\mathbb{C}}$$

$$\mathsf{AFiniteSch}_{\mathbb{C}}^{\mathsf{op}} \cong \mathsf{FGCAlg}_{\mathbb{C}}$$

Lemma 2.20. *Let $\phi : R \to S$ be a surjective $\mathbb{C}$-algebra homomorphisms between finite $\mathbb{C}$-algebras, then $\operatorname{Spec}\phi : \operatorname{Spec} S \to \operatorname{Spec} R$ is an embedding.*

Proof. $\operatorname{Spec}\phi$ is injective since ϕ is surjective. We need to check that every open set in $\operatorname{Spec} S$ is an inverse image of some open set in $\operatorname{Spec} R$. Since $\operatorname{Spec} S$ has a basis of basic open sets $D(f)$'s where $f \in S$, we only need to check $D(f)$ is the inverse image of some open set in $\operatorname{Spec} R$. Since θ is surjective, we know that $\theta(r) = f$ for some $r \in R$. Then $D(f) \subseteq \operatorname{Spec} S$ is the inverse image of $D(r) \subseteq \operatorname{Spec} R$. □

[1] finite union of quasicompact set is again quasicompact

2.3 Closed points

In this section, we focus on the subset set of closed points of a scheme locally of finite type over $\mathbb{C}$, preparing for the complex topology. Let us fix some notation first: for an arbitrary scheme $(X, \mathcal{O}_X)$, we denote $\operatorname{Max}\{X\}$ to be the set of all closed points of X and $\operatorname{MaxSpec} R$ to be the set of all maximal ideals of a ring R. Note that $\operatorname{MaxSpec} R$ is exactly $\operatorname{Max}\{\operatorname{Spec} R\}$ and we use both interchangeably.

Proposition 2.21. *Let $\left(\operatorname{Spec} A, \widetilde{A}\right)$ be an affine finite scheme over $\mathbb{C}$. We have that the set of closed points $\operatorname{MaxSpec} A$ are in bijection to $\operatorname{Hom}_{\mathsf{CAlg}_{\mathbb{C}}}(A, \mathbb{C})$*

Proof. From Corollary 1.3, we know that for each closed point $\mathfrak{m}$, i.e. a maximal ideal, there is a unique $\phi_{\mathfrak{m}} : A \to \mathbb{C}$ whose kernel is $\mathfrak{m}$. Conversely, for any $\phi : A \to \mathbb{C}$, $\ker \phi$ is certainly a prime ideal[2]. Since ϕ is surjective[3], its kernel is maximal.

It remains to show that $\mathfrak{m} \mapsto \phi_{\mathfrak{m}}$ and $\phi \mapsto \ker \phi$ are inverse to each other. But this follows from the uniqueness from Corollary 1.3: Let $\mathfrak{m}$ be a maximal ideal, then the $\ker \phi_{\mathfrak{m}}$ is exactly $\mathfrak{m}$ by definition of $\phi_{\mathfrak{m}}$; on the other hand, if ϕ is an algebra homomorphism then ϕ and $\phi_{\ker \phi}$ are both algebra homomorphism that has kernel $\ker \phi$, hence must be equal. $\square$

Corollary 2.22. *Let $\left(\operatorname{Spec} A, \widetilde{A}\right)$ be an affine finite scheme over $\mathbb{C}$. We have that the* $\operatorname{MaxSpec} A$ *is in bijection with* $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{C}}\left(\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right), \left(\operatorname{Spec} A, \widetilde{A}\right)\right)$

Proposition 2.23. *If $(X, \mathcal{O}_X)$ is a scheme locally of finite type over $\mathbb{C}$, then* $\operatorname{Max}\{X\}$ *is in bijection with $X(\mathbb{C}) := \operatorname{Hom}_{\mathsf{Sch}/\mathbb{C}}\left(\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right), (X, \mathcal{O}_X)\right)$, such that every closed point p, is the image of $\star$ of a unique morphism Φ_p; and for each morphism $\Phi : \left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) \to (X, \mathcal{O}_X)$, $\Phi(\star)$ is closed in X where $\star$ is the unique point of* $\operatorname{Spec} \mathbb{C}$.

Proof. Let $x \in X$ be a closed point and an affine open neighbourhood of $x \in U \cong \left(\operatorname{Spec} A, \widetilde{A}\right)$ where A is a finite $\mathbb{C}$-algebra. Thus the x corresponds to a morphism Φ_A between $\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right)$ and $\left(\operatorname{Spec} A, \widetilde{A}\right)$ by Corollary 2.22; we define Ψ_x to be the composition of

$$\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) \xrightarrow{\Phi_A} \left(\operatorname{Spec} A, \widetilde{A}\right) \xrightarrow{\sim} (U, \mathcal{O}_X|_U) \hookrightarrow (X, \mathcal{O}_X).$$

Moreover, Ψ_x does not dependent on the choice of affine neighbourhood $\operatorname{Spec} A$: suppose $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$, then $\operatorname{Spec} A \cap \operatorname{Spec} B$ admits an open covering of spectra of finitely generated $\mathbb{C}$-algebras by Proposition 2.13. Thus we can find a finitely generated $\mathbb{C}$-algebra C such that $\operatorname{Spec} C \subseteq \operatorname{Spec} A \cap \operatorname{Spec} B$.

$$\begin{array}{ccccccc}
 & & & & \left(\operatorname{Spec} A, \widetilde{A}\right) & & \\
 & & & \nearrow & & \searrow & \\
\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) & \xrightarrow{\Phi_C} & \left(\operatorname{Spec} C, \widetilde{C}\right) & & & & (X, \mathcal{O}_X), \\
 & & & \searrow & & \nearrow & \\
 & & & & \left(\operatorname{Spec} B, \widetilde{B}\right) & &
\end{array}$$

[2] $\ker \phi$ is equal to $(\operatorname{Spec} \phi)(\star)$ where $\star$ is the unique point of $\operatorname{Spec} \mathbb{C}$

[3] for each $c \in \mathbb{C}$, $\phi(c \cdot 1) = c$

where $(\_ : \operatorname{Spec} C \hookrightarrow \operatorname{Spec} A) \circ \Phi_C$ is exactly Φ_A and $(\_ : \operatorname{Spec} C \hookrightarrow \operatorname{Spec} B) \circ \Phi_C$ is exactly Φ_B by Corollary 2.22; thus both composition in the commutative square above is Ψ_x, in another word, Ψ_x is independent from the choice of affine neighbourhood.

On the otherhand, if we are given a morphism $\Psi : \left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) \to \left(\operatorname{Spec} A, \widetilde{A}\right)$, let us denote x to be the image of the unique point in $\operatorname{Spec} \mathbb{C}$ under Ψ; we want to show that x is a closed point. Since affine open set forms a basis, we only need to check that, for any affine open $\left(\operatorname{Spec} A, \widetilde{A}\right) \hookrightarrow (X, \mathcal{O}_X)$, x is closed in $\operatorname{Spec} A$. We consider the factorisation of Ψ:

$$\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) \xrightarrow{\operatorname{Spec} \psi} \left(\operatorname{Spec} A, \widetilde{A}\right) \hookrightarrow (X, \mathcal{O}_X),$$

where ψ is a $\mathbb{C}$-algebra homomorphism $A \to \mathbb{C}$ such that $\operatorname{Spec} \psi = \Psi|_{\operatorname{Spec} A}$, hence by Corollary 2.22, we have x is closed in $\left(\operatorname{Spec} A, \widetilde{A}\right)$. The two construction above is bijection is verified as the following:

1. Let x be a closed point, then it corresponds to Ψ_x, but the image of the unique point in $\operatorname{Spec} \mathbb{C}$ under Ψ_x is x;
2. if Φ is a morphism $\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) \to X$ and denote the unique image as x, Φ factors through affine open neighbourhood of x hence it is Ψ_x because Ψ_x does not dependent on the choice of affine neightbourhood.

□

Proposition 2.24. *Let $\Phi = (\phi, \phi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes locally of finite type over $\mathbb{C}$, then ϕ maps closed points of X to closed points of Y. Thus, we have a well defined map $\operatorname{Max}\{\phi\} : \operatorname{Max}\{X\} \to \operatorname{Max}\{Y\}$*

Proof. Let x be a closed point in X, then x corresponds to a unique $\Psi_x = (\psi_x, \psi_x^*)$: $\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) \to X$ such that $\psi_x(\star) = x$ where $\star$ is the unique point of $\operatorname{Spec} \mathbb{C}$. The composite $\Phi \circ \Psi_x$ is a morphism $\left(\operatorname{Spec} \mathbb{C}, \widetilde{\mathbb{C}}\right) \to Y$ thus $\Phi \circ \Psi_x(\star)$ is closed in Y, since $\Phi \circ \Psi_x(\star) = \Phi(\Psi_x(\star)) = \Phi(x)$, we conclude that $\Phi(x)$ is a closed point in Y. □

When we talk about topology on $\operatorname{Max}\{X\}$, we mean the subspace topology induced by the Zariski topology[4].

Corollary 2.25. *Let $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an open immersion, then $\operatorname{Max}\{\phi\}$ is an embedding.*

Proof. Write $\Phi = (\phi, \phi^*)$, note that ϕ is necessarily an embedding $X \hookrightarrow Y$ thus $\operatorname{Max}\{\phi\}$ being the restriction of ϕ must be embedding as well:

$$\begin{array}{ccc} \operatorname{Max}\{X\} & \xrightarrow{\operatorname{Max}\{\phi\}} & \operatorname{Max}\{Y\} \\ \downarrow & & \downarrow \\ X & \xhookrightarrow{\phi} & Y \end{array}$$

where the vertical arrows and ϕ are embeddings so $\operatorname{Max}\{\phi\}$ is embedding as well. □

By the same argument and using Lemma 2.20, we can prove the following lemma

Lemma 2.26. *Let $\theta : R \to S$ be a surjective $\mathbb{C}$-algebra homomorphism between finite $\mathbb{C}$-algebras. $\operatorname{MaxSpec} \theta$ is an embedding.*

[4] Hopefully, I will be able to main the notation clearly: $\operatorname{Max}\{X\}$ is Zariski and $\{X\}^{\mathsf{an}}$ is analytic

Remark 2.27. *Let X be a scheme locally of finite type over $\mathbb{C}$ and $\mathcal{U} = \{U_i\}$ be an open cover of X. Then* $\operatorname{Max}\{X\} = \bigcup_i \operatorname{Max}\{U_i\}$*, so that* $\operatorname{Max}\{\mathcal{U}\} = \{\operatorname{Max}\{U_i\}\}$ *is a Zariski open cover for* $\operatorname{Max}\{X\}$

Lemma 2.28. *Let $\theta : R \to S$ be a surjective $\mathbb{C}$-algebra homomorphism between finite $\mathbb{C}$-algebras. Then the image of* $\operatorname{MaxSpec}\theta : \operatorname{MaxSpec} S \to \operatorname{MaxSpec} R$ *is identified via Proposition 2.21 with the set of $\mathbb{C}$-algebra homomorphisms $\psi : R \to \mathbb{C}$ such that* $\psi(\ker\theta) = 0$.

Proof. Let $\mathfrak{m} \subseteq R$ be a maximal ideal inside the image of $\operatorname{MaxSpec}\theta$, i.e. there exists a maximal ideal $\mathfrak{p} \subseteq S$ such that $\theta^{-1}\mathfrak{p} = \mathfrak{m}$. $\mathfrak{m}$ corresponds to the unique algebra homomorphism $\phi_{\mathfrak{m}} : R \to \mathbb{C}$ whose kernel is $\mathfrak{m}$ and $\mathfrak{p}$ corresponds to the unique algebra homorphism $\psi_{\mathfrak{p}} : S \to \mathbb{C}$ whose kernel is $\mathfrak{p}$. Thus $\theta^{-1}\mathfrak{p} = \mathfrak{m}$ precesily when $\psi_{\mathfrak{p}} \circ \theta = \phi_{\mathfrak{m}}$; and this happens precisely when $\psi_{\mathfrak{p}}$ annaliates the kernel of θ. □

Remark 2.29. *If we are only considering schemes (locally of) finite type over $\mathbb{C}$, any morphism of ringed space over $\mathbb{C}$ is automatically a morphism of* locally *ringed space over $\mathbb{C}$.*

Chapter 3

Analytification of a scheme

3.1 Toplogical story

3.1.1 Affine scheme

Let S be a finitely generated $\mathbb{C}$-algebra so that $S \cong \mathbb{C}[a_1, \dots, a_n]$ for some $a_i \in S$. Thus there is a surjection $\theta : \mathbb{C}[X_1, \dots, X_n] \to S$ defined by $X_i \mapsto a_i$. Thus, we have a morphism $\left(\operatorname{Spec}\theta, \tilde{\theta}\right)$ of schemes of finite type over $\mathbb{C}$ between $\left(\operatorname{Spec} S, \tilde{S}\right)$ to $\left(\operatorname{Spec}\mathbb{C}[X_1, \dots, X_n], \widetilde{\mathbb{C}[X_1, \dots, X_n]}\right)$. By Proposition 2.24, we know that $\operatorname{Spec}\theta$ gives us a continuous map

$$\operatorname{MaxSpec}\theta : \operatorname{MaxSpec} S \to \operatorname{MaxSpec}\mathbb{C}[X_1, \dots, X_n],$$

since θ is surjective, $\operatorname{MaxSpec}\theta$ is injective[1].

Theorem 3.1. *The set of closed points in* $\operatorname{Spec}\mathbb{C}[X_1, \dots, X_n]$ *corresponds bijectively to* $\mathbb{C}^n$.

Proof. By Proposition 2.21, the set of closed points bijects to $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}$. Thus we only need a bijection between $\mathbb{C}$-algebra homomorphism $\mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}$ and $\mathbb{C}^n$:

1. Give a point $p := (a_1, \dots, a_n) \in \mathbb{C}^n$, we define $\phi_a : \mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}$ to be evaluation at the point p.
2. Give a $\mathbb{C}$-algebra homomorphism ϕ, we take the point to be $(\phi(X_1, \dots, X_n))$.

□

Definition 3.2 (Analytification of topological spaces.)**.** *The complex topology of* $\operatorname{Spec} S$ *is the subspace topology of* $\mathbb{C}^n$ *via the injective map* $\operatorname{MaxSpec}\theta$. *With the complex topology, we denote* $\operatorname{MaxSpec} S$ *as* $\{(\operatorname{Spec} S)\}^{\mathsf{an}}$ *and, if* $\phi : S \to S'$ *is a* $\mathbb{C}$*-algebra homorphism, we the induced map between* $\{\operatorname{Spec} S'\}^{\mathsf{an}}$ *and* $\{\operatorname{Spec} S\}^{\mathsf{an}}$ *as* $\{\operatorname{Spec}\phi\}^{\mathsf{an}}$.

Note that by now we do not know that $\{(\operatorname{Spec} S)\}^{\mathsf{an}}$ is independent from the choice of generators $\{a_1, \dots, a_n\}$, we will enventually prove that this is true, but let's write $\{(\operatorname{Spec} S)\}^{\mathsf{an}}_{a_i}$ to stress the dependency.

Theorem 3.3. *If* S*, as* $\mathbb{C}$*-algebras, is generated by both* $a_1, \dots, a_n$ *and* $b_1, \dots, b_m$*, we would have as topological spaces* $\{(\operatorname{Spec} S)\}^{\mathsf{an}}_{a_i}$ *and* $\{(\operatorname{Spec} S)\}^{\mathsf{an}}_{b_i}$ *are homeomophic.*

[1] being the restriction of the injective function $\operatorname{Spec}\theta$

Proof. Let us abbreviate the polynomial rings $\mathbb{C}[X_1, \dots, X_n]$ as R and $\mathbb{C}[Y_1, \dots, Y_m]$ as R', then we have two surjective homomorphisms $\theta : R \to S$ and $\theta' : R' \to S$ such that $\theta(X_i) = a_i$ and $\theta'(Y_i) = b_i$.

It is sufficient, by symmetry, to prove topology induced by generators b_i's is finer than that of a_i's.

Since b_i's generate S and a_i's are in S, we can find n polynomials $P_i \in R' = \mathbb{C}[Y_1, \dots, Y_m]$ such that $a_i = P_i(b_1, \dots, b_m)$. Thus we can define $\phi : R \to R'$ by $X_i \mapsto P_i(Y_1, \dots, Y_m)$ such that $\theta = \theta' \circ \phi$. Thus we have a commutative diagram (of plain functions)

$$\begin{array}{ccc}
\{(\operatorname{Spec} S)\}^{\mathsf{an}}_{a_i} & \xrightarrow{\operatorname{MaxSpec}\theta} & \mathbb{C}^n \\
{\scriptstyle =}\updownarrow & & \uparrow{\scriptstyle \operatorname{MaxSpec}\phi} \\
\{(\operatorname{Spec} S)\}^{\mathsf{an}}_{b_i} & \xrightarrow{\operatorname{MaxSpec}\theta'} & \mathbb{C}^m
\end{array}$$

It is sufficient to prove that $\operatorname{MaxSpec}\phi$ is continuous, then since $\operatorname{MaxSpec}\theta$, $\operatorname{MaxSpec}\theta'$ and $\operatorname{MaxSpec}\phi$ are all continuous, the identity function $\{(\operatorname{Spec} S)\}^{\mathsf{an}}_{a_i} \to \{(\operatorname{Spec} S)\}^{\mathsf{an}}_{b_i}$ is continuous. Consider a point $c = (c_1, \dots, c_m) \in \mathbb{C}^m$, then $\operatorname{MaxSpec}\phi(c)$ is the point $\operatorname{eval}_c \circ \phi(X_1, \dots, X_n)$, i.e. $(c_1, \dots, c_m) \mapsto (P_1(c_1, \dots, c_m), \dots, P_n(c_1, \dots, c_m))$. This is a map defined by polynomials, thus is continuous. □

Now we have proven that the complex topology is independent of generators, we can write $\{(\operatorname{Spec} S)\}^{\mathsf{an}}$ with a clear conscience.

Lemma 3.4. *Since* $\{(\operatorname{Spec} S)\}^{\mathsf{an}}$*, as a set, is just the set of closed points of* $\operatorname{Spec} S$*, we have a function* $\lambda : \{(\operatorname{Spec} S)\}^{\mathsf{an}} \hookrightarrow \operatorname{Spec} S$*.* λ *is continuous where* $\{(\operatorname{Spec} S)\}^{\mathsf{an}}$ *is with complex topology while* $\operatorname{Spec} S$ *is with the Zaraski toplogy.*

Proof. Let us choose a set of generators $a_1, \dots, a_n$ and write $R := \mathbb{C}[X_1, \dots, X_n]$, then we would have the following commutative diagram:

$$\begin{array}{ccc}
\{(\operatorname{Spec} S)\}^{\mathsf{an}} & \xhookrightarrow{\lambda_S} & \operatorname{Spec} S \\
{\scriptstyle \text{restriction of } \operatorname{Spec}\theta}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}\theta} \\
\{(\operatorname{Spec} R)\}^{\mathsf{an}} & \xhookrightarrow{\lambda_R} & \operatorname{Spec} R \\
{\scriptstyle \sim}\downarrow & & \\
\mathbb{C}^n & &
\end{array},$$

where θ is the surjective $\mathbb{C}$-algebra homomorphism $R \to S$. The red arrows are continuous, since they define the complex topology; $\operatorname{Spec}\theta$ is continuous as well. To prove λ_S is continuous, we only need to prove the special case λ_R where $R = \mathbb{C}[X_1, \dots, X_n]$. Since $\operatorname{Spec} R$ has a basis of basic open set $D(f)$, we only need to check that $D(f) \cap \{(\operatorname{Spec} R)\}^{\mathsf{an}}$ is open for any polynomial $f \in \mathbb{C}[X_1, \dots, X_n]$, indeed the intersection is equal to $\{x \in \mathbb{C}^n | f(x) \neq 0\}$ thus open[2][3].

□

[2] $\mathfrak{p} \in D(f)$ if and only if $f \notin \mathfrak{p}$. Hence $\mathfrak{m}$ is in the intersection if and onlyv if $\mathfrak{m}$ is equal to the kernel of evaluation map ϕ_a at some point a and that f is not in the kernel, in another word, $f(a) \neq 0$.

[3] should this be a separate lemma?

Lemma 3.5. *Let $a_1, \dots, a_n$ be a set of generators of S as $\mathbb{C}$-algebra and R be $\mathbb{C}[X_1, \dots, X_n]$ and $\theta : R \to S$ be the surjective map defined by $\theta(X_i) = a_i$. The image of* $\operatorname{Spec}\theta : \{\operatorname{Spec} S\}^{\mathsf{an}} \to \{\operatorname{Spec} R\}^{\mathsf{an}} \cong \mathbb{C}^n$ *is*

$$V(\ker\theta) := \{(x_1, \dots, x_m) | f_i(x_1, \dots, x_n) = 0\} = \{(x_1, \dots, x_m) | p(x_1, \dots, x_n) = 0 \textit{ for all } p \in \ker\theta\},$$

where f_i generates $\ker\theta$.

Proof. $x = (x_1, \dots, x_n) \in \operatorname{image}\{\operatorname{Spec}\theta\}^{\mathsf{an}}$ if and only if ψ_x, evaluation at x, annilates the kernel of θ by Lemma 2.28 □

Theorem 3.6. *Let S and S' be two finitely generated $\mathbb{C}$-algebras and $\phi : S \to S'$ be a $\mathbb{C}$-algebra homomorphism, the natural map* $\{\operatorname{Spec}\phi\}^{\mathsf{an}} : \{\operatorname{Spec} S'\}^{\mathsf{an}} \to \{\operatorname{Spec} S\}^{\mathsf{an}}$ *is continuous (in the complex topology) and compatible with the inclusion map, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} \{\operatorname{Spec} S'\}^{\mathsf{an}} & \xrightarrow{\{\operatorname{Spec}\phi\}^{\mathsf{an}}} & \{\operatorname{Spec} S\}^{\mathsf{an}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} S' & \xrightarrow{\operatorname{Spec}\phi} & \operatorname{Spec} S \end{array}$$

Proof. The commutativity is free. Let us choose generators $\{a_1, \dots, a_n\}$'s for S and $\{b_1, \dots, a_m\}$'s for S'. Let us write the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$ as R and $\mathbb{C}[Y_1, \dots, Y_m]$ as R'. Then we have two surjective $\mathbb{C}$-algebra homomorphisms $\theta : R \to S$ and $\theta' : R' \to S'$ as usual. Since $\{b_i\}$ generates S', we can find polynomials $P_i \in R'$ such that $\phi(a_i) = P_i(b_1, \dots, b_m)$. Then we can define a $\mathbb{C}$-algebra homomorphism $\psi : R \to R'$ by $X_i \mapsto P_i(Y_1, \dots, Y_m)$ giving us the following commutative diagrams:

$$\begin{array}{ccc} R & \xrightarrow{\psi} & R' \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} \\ S & \xrightarrow{\phi} & S' \end{array} \qquad \begin{array}{ccc} \mathbb{C}^n & & \mathbb{C}^m \\ \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\ \{\operatorname{Spec} R\}^{\mathsf{an}} & \xleftarrow{\{\operatorname{Spec}\psi\}^{\mathsf{an}}} & \{\operatorname{Spec} R'\}^{\mathsf{an}} \\ {\scriptstyle\{\operatorname{Spec}\theta\}^{\mathsf{an}}}\uparrow & & {\scriptstyle\operatorname{Spec}\theta'}\uparrow \\ \{\operatorname{Spec} S\}^{\mathsf{an}} & \xleftarrow[\{\operatorname{Spec}\phi\}^{\mathsf{an}}]{} & \{\operatorname{Spec} S'\}^{\mathsf{an}} \end{array}$$

The red arrows are continuous because they define the complex topology and $\{\operatorname{Spec}\psi\}^{\mathsf{an}}$ is continuous because it is defined by polynomial ψ. Thus $\{\operatorname{Spec}\phi\}^{\mathsf{an}}$ is continuous. □

Corollary 3.7. *If $S \to S'$ is an isomorphism of finite $\mathbb{C}$-algebras, then* $\{\operatorname{Spec} S\}^{\mathsf{an}}$ *and* $\{\operatorname{Spec} S'\}^{\mathsf{an}}$ *are homeomorphic.*

Lemma 3.8. *If $\phi : S \to S'$ is a surjective $\mathbb{C}$-algebra homomorphism between two finite $\mathbb{C}$-algebras, then* $\{\operatorname{Spec}\phi\}^{\mathsf{an}} : \{\operatorname{Spec} S'\}^{\mathsf{an}} \to \{\operatorname{Spec} S\}^{\mathsf{an}}$ *is an embedding.*

Proof. Let $\{a_1, \dots, a_n\}$'s be generators of S and R be the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. Then we have $\theta : R \to S$ such that $\theta(X_i) = a_i$. The composition $R \xrightarrow{\theta} S \xrightarrow{\phi} S'$ is a surjection as well. Thus by taking $\{\operatorname{Spec}(-)\}^{\mathsf{an}}$ operation, we get

$$\{\operatorname{Spec} S'\}^{\mathsf{an}} \xrightarrow{\{\operatorname{Spec}\phi\}^{\mathsf{an}}} \{\operatorname{Spec} S\}^{\mathsf{an}} \xrightarrow{\{\operatorname{Spec}\theta\}^{\mathsf{an}}} \{\operatorname{Spec} R\}^{\mathsf{an}} \xrightarrow{\sim} \mathbb{C}^n.$$

The whole composition is embedding because of independence of generators and $\{\operatorname{Spec}\theta\}^{\mathsf{an}}$ is an embedding as well, thus $\{\operatorname{Spec}\phi\}^{\mathsf{an}}$ is an embedding as well. □

Lemma 3.9. *Let us write $\mathbb{C}[X_1, \dots, X_n]$ as R and let $f \in R$ be a polynomial, then the localization map $\alpha : R \to R_f$ induces an embedding $\left\{\operatorname{Spec} R_f\right\}^{\mathsf{an}} \hookrightarrow \{\operatorname{Spec} R\}^{\mathsf{an}}$.*

Proof. TBD □

More generally, we have a corresponding lemma for arbitrary finite $\mathbb{C}$-algebras.

Lemma 3.10. *If S is a finite $\mathbb{C}$-algebra and $f \in S$, then the localization map $\alpha : S \to S_f$ induces an embedding $\left\{\operatorname{Spec} S_f\right\}^{\mathsf{an}} \hookrightarrow \{\operatorname{Spec} S\}^{\mathsf{an}}$. In fact $\left\{\operatorname{Spec} S_f\right\}^{\mathsf{an}}$ is identified as the subset $D(f) \cap \operatorname{Spec} S$ where $D(f)$ is the basic open set in $\operatorname{Spec} S$.*

Proof. TBD □

3.1.2 Arbitrary scheme

Definition 3.11 (Complex Topology)**.** *Let $(X, \mathcal{O}_X)$ be a scheme locally of finite type over $\mathbb{C}$, let $\mathcal{I}$ be the collection of open immersions $\left(\operatorname{Spec} R, \widetilde{R}\right) \to X$ where R is some finite $\mathbb{C}$-algebra. Then the complex topology on the set of closed points $\max X$ is defined as the weak topology with respect to $\left\{\{\phi\}^{\mathsf{an}} | (\phi, \phi^*) \in \mathcal{I}\right\}$ where $\{\phi\}^{\mathsf{an}}$ is the restriction of ϕ to the subset of closed points. When we talk about complex toplogy, we write $\max X$ as $\{X\}^{\mathsf{an}}$.*

Lemma 3.12. *Let $(X, \mathcal{O}_X)$ be a scheme locally of finite type over $\mathbb{C}$, R be a finite $\mathbb{C}$-algebra and $\Psi = (\psi, \psi^*) : \left(\operatorname{Spec} R, \widetilde{R}\right) \to (X, \mathcal{O}_X)$ be an open immersion. Then $\{\psi\}^{\mathsf{an}}$ is an embedding.*

Proof. TBD □

Lemma 3.13. *Let $\lambda_X : \{X\}^{\mathsf{an}} \hookrightarrow X$ be the inclusion map, then λ_X is continuous.*

Proof. Let $x \in X$ and $U \cong \operatorname{Spec} A$ be an open affine neighbourhood of x. Then we have the following commutative diagram:

$$\begin{array}{ccc} \{\operatorname{Spec} A\}^{\mathsf{an}} & \xrightarrow{\iota_1} & \{X\}^{\mathsf{an}} \\ \big\downarrow & & \big\downarrow \lambda_X \\ \operatorname{Spec} A & \xhookrightarrow{\iota_2} & X. \end{array}$$

The red arrow is continuous by Lemma 3.4; ι_1 is continuous by Lemma 3.12; ι_2 is an open embedding by hypothesis. Thus λ_X is continuous as well. □

Lemma 3.14. *Let $\Phi = (\phi, \phi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes locally of finite type over $\mathbb{C}$, then $\{\phi\}^{\mathsf{an}} : \{X\}^{\mathsf{an}} \to \{Y\}^{\mathsf{an}}$ is continuous.*

Proof. Let $x \in X$ and $U \cong \operatorname{Spec} A \subseteq X$ and $V \cong \operatorname{Spec} B \subseteq Y$ such that $\phi U \subseteq V$ be affine neighbourhoods around x and $\phi(x)$[4]. Then $\Phi|_U : \left(\operatorname{Spec} A, \widetilde{A}\right) \to \left(\operatorname{Spec} B, \widetilde{B}\right)$ is induced by a $\mathbb{C}$-algebra homomorphism $\alpha : B \to A$, thus we have the following two commutative squares:

[4]This is possible by definition of being locally finite.

$$\begin{array}{ccc} (\operatorname{Spec} A, \widetilde{A}) & \xrightarrow{(\operatorname{Spec} \alpha, \tilde{\alpha})} & (\operatorname{Spec} B, \widetilde{B}) \\ \downarrow & & \downarrow \\ (X, \mathcal{O}_X) & \xrightarrow{\Phi} & (Y, \mathcal{O}_Y) \end{array} \qquad \begin{array}{ccc} \{\operatorname{Spec} A\}^{\mathsf{an}} & \xrightarrow{\{\operatorname{Spec} \alpha\}^{\mathsf{an}}} & \{\operatorname{Spec} B\}^{\mathsf{an}} \\ \downarrow & & \downarrow \\ \{X\}^{\mathsf{an}} & \xrightarrow{\{\phi\}^{\mathsf{an}}} & \{Y\}^{\mathsf{an}}. \end{array}$$

where the red arrows are continuous by Lemma 3.13 and $\{\operatorname{Spec} \alpha\}^{\mathsf{an}}$ is continuous by Theorem 3.6. Thus $\{\phi\}^{\mathsf{an}}$ is continuous as well. □

Corollary 3.15. *Let $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an open immersion of schemes locally of finite type over $\mathbb{C}$, then $\{\phi\}^{\mathsf{an}}$ is an embedding of $\{X\}^{\mathsf{an}}$ into $\{Y\}^{\mathsf{an}}$.*

Proof. TBD □

Here are two easy consequences:

Corollary 3.16. *Given two morphisms among schemes locally of finite type over $\mathbb{C}$*

$$(X, \mathcal{O}_X) \xrightarrow{(\phi, \phi^*)} (Y, \mathcal{O}_Y) \xrightarrow{(\psi, \psi^*)} (Z, \mathcal{O}_Z),$$

we have $\{\psi \circ \phi\}^{\mathsf{an}} = \{\psi\}^{\mathsf{an}} \circ \{\phi\}^{\mathsf{an}}$

Proof. Restriction of composition is composition of restriction. □

Corollary 3.17. *Let $\Phi = (\phi, \phi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes locally of finite type over $\mathbb{C}$, we have*

$$\begin{array}{ccc} \{X\}^{\mathsf{an}} & \xrightarrow{\{\phi\}^{\mathsf{an}}} & \{Y\}^{\mathsf{an}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & Y \end{array}$$

3.2 Sheaf side

In previous section, we have showed that for any scheme $(X, \mathcal{O}_X)$ locally of finite type over $\mathbb{C}$, we can make a topological space $\{X\}^{\mathsf{an}}$ with the complex topology. In this section, we aim to make a sheaf $\{\mathcal{O}_X\}^{\mathsf{an}}$ on $\{X\}^{\mathsf{an}}$. We will consider the affine cases $(\operatorname{Spec} S, \tilde{S})$ where S is a finite $\mathbb{C}$-algebra by choosing generators; then we prove that the construction does not dependent on the choice of generators; then we glue everything together.

3.2.1 Affine schemes

Let S be a finite $\mathbb{C}$-algebra, we choose generators $a_1, \dots, a_n$ and write $\mathbb{C}[X_1, \dots, X_n]$ as R. Thus we have a surjective $\mathbb{C}$-algebra homomorphism $\theta : R \to S$ such that $\theta(X_i) = a_i$. Then $\ker \theta$ is finitely generated as well, say by $f_1, \dots, f_m$. Note that the image of $\{\operatorname{Spec} \theta\}^{\mathsf{an}}$: $\{\operatorname{Spec} S\}^{\mathsf{an}} \to \{\operatorname{Spec} \mathbb{C}[X_1, \dots, X_n]\}^{\mathsf{an}} \cong \mathbb{C}^n$, which we often just write as $\{\operatorname{Spec} \theta\}^{\mathsf{an}}$ as well, is in bijection with $V(\ker \theta) := \{x \in \mathbb{C}^n \mid f_i(x) = 0 \text{ for all } i\}$[5]; hence is closed in $\mathbb{C}^n$ because its the intersection of inverse images of the singleton set $\{0\}$.

We do constructions using $f_1, \dots, f_m$, so everything depends on the choice of $a_1, \dots, a_n$. In this section, we also denote the sheaf of holomorphic function as $\mathcal{H}ol$ where $\Gamma(\mathcal{H}ol, U) := \{f : U \to \mathbb{C} | f \text{ is holomorphic}\}$ for any open subset $U \subseteq \mathbb{C}^n$.

We use a specialized basis of topology on $\mathbb{C}^n$.

[5] by Lemma 2.28

Definition 3.18 (Generalized polydiscs)**.** *A generalized polydisc*

$$\Delta(g_1, \dots, g_l; w_1, \dots, w_l; r_1, \dots, r_l)$$

is the set $\{x \in \mathbb{C}^n | |g_i(x) - w_i| < r_i$ *for all* $i = 1, \dots, l\}$ *where each* $g_i \in \mathbb{C}[X_1 \dots, X_n]$ *is a polynomial of* n *variables,* $w := (w_1, \dots, w_n) \in \mathbb{C}^l$ *is a point and* $r := (r_1, \dots, r_l) \in \mathbb{R}^l_{\geq 0}$ *are all non-negative. We call* w *the center of the polydisc and* r *the polyradius*[6]*.*

Remark 3.19. *Traditionally, a polydisc* $\Delta(w_1, \dots, w_l; r_1, \dots, r_l) := \{x \in \mathbb{C}^l | |w_i - r_i| < r_i\}\}$ *is the special case* $\Delta(X_1, \dots, X_n; w_1, \dots, w_n; r_1, \dots, r_n)$*. For a generalized polydisc* $\Delta(g; w; r)$*, we have a map* $g : \mathbb{C}^n \to \mathbb{C}^l$ *defined by* $x \mapsto (g_1(x), \dots, g_l(x))$*. Note that* $\Delta(g; w; r)$ *is equal to* $g^{-1}\Delta(w; r)$*, the inverse image of the usual polydisc. So generalized polydiscs are exactly the inverse image of usual polydiscs by some polynomial map.*

Lemma 3.20 (Basis of generalized polydiscs)**.** *All generalized polydiscs form a topological basis of* $\mathbb{C}^n$*.*

Proof.

- Every open set is a union of generalized polydiscs. Just take small polyradius and $g_i = X_i$.
- Intersection of two generalized polydiscs is a generalized polydisc. Consider $\Delta(g; w; r)$ and $\Delta(g'; w'; r')$. The idea is the following: $\Delta(g; w; r)$ is $g^{-1}\Delta(w; r)$ and $\Delta(g'; w'; r') = {g'}^{-1}\Delta(g'; w; r'))$, and $(g, g') : \mathbb{C}^n \to \mathbb{C}^l \times \mathbb{C}^{l'} \cong \mathbb{C}^{l+l'}$ is a polynomial map, call it G. Since $\Delta(w; r) \times \Delta(w'; r')$ is a usual polydisc, its inverse image under G is a generalized polydiscs and is equal to the intersection of the original generalized polydiscs.

□

The following terminology is nonstandard.

Definition 3.21 (Preanalytification of sheaves.)**.** *We have a sheaf* $\mathcal{O}^{\mathsf{pre}}$ *on* $\mathbb{C}^n$ *is defined to be the cokernel sheaf of the following exact sequence:*

$$\mathcal{H}ol^{\oplus m} \longrightarrow \mathcal{H}ol \longrightarrow \mathcal{O}^{\mathsf{pre}} \longrightarrow 0,$$

where the first arrow, on U*, is given by* $(a_1, \dots, a_m) \mapsto f_1|_U \cdot a_1 + \dots + f_m|_U \cdot a_m$ *where* $a_i \in \Gamma(\mathcal{H}ol, U)$ *and* f_i *generates* $\ker\theta$

Lemma 3.22. *On generalized polydiscs* $\Delta_1 \subseteq \Delta_2$*,* $\Gamma(\mathcal{O}^{\mathsf{pre}}, \Delta_1)$ *is isomorphic to*

$$\Gamma(\mathcal{H}ol, \Delta_1) \big/ \ker\theta \cdot \Gamma(\mathcal{H}ol, \Delta_1)$$

and the restriction map $\Gamma(\mathcal{O}^{\mathsf{pre}}, \Delta_2) \to \Gamma(\mathcal{O}^{\mathsf{pre}}, \Delta_1)$ *is the restriction map*

$$\mathrm{res} : \Gamma(\mathcal{H}ol, \Delta_2) \big/ \ker\theta \cdot \Gamma(\mathcal{H}ol, \Delta_2) \to \Gamma(\mathcal{H}ol, \Delta_1) \big/ \ker\theta \cdot \Gamma(\mathcal{H}ol, \Delta_1)$$

defined by $[f] \mapsto [f|_U]$*.*

Proof. This lemma is going to be hard. Uses coherent analytic sheaves, Cartan's Theorem, Stein manifold, etc. The author of [2, page 108] tells us to look at [1, page 136, definition 2; page 243, theorem 2]. □

[6] Different polydiscs can have different l

Lemma 3.23. *Let x be a point not in $\{\operatorname{Spec} S\}^{\mathsf{an}}$, in another word, $x \in \mathbb{C}^n - \{\operatorname{Spec} S\}^{\mathsf{an}}$. There is an open subset $x \in U \subseteq \mathbb{C}^n$ such that for any generalized polydiscs $\Delta \subseteq U$, we have $\Gamma(\mathcal{O}^{\mathsf{pre}}, \Delta) = 0$.*

Proof. By Lemma 2.28, x is not in the image of $\{\operatorname{Spec} S\}^{\mathsf{an}}$ precisely when at least one of the $f_i(x) \neq 0$. Say it's the first one. Then, there is some open neighbourhood U around x such that $f_1 |_U$ is nowhere zero, i.e. invertible. Thus for any generalized polydisc $\Delta \subseteq U$, we have that the ideal $\ker\theta \cdot \Gamma(\mathcal{H}ol, U)$ is everything so that $\Gamma(\mathcal{O}^{pre}, \Delta) \cong \Gamma(\mathcal{H}ol, \Delta) \big/ \ker\theta \cdot \Gamma(\mathcal{H}ol, \Delta)$ is trivial. □

Corollary 3.24 ($\mathcal{O}^{\mathsf{pre}}$ is supported in $\{\operatorname{Spec} S\}^{\mathsf{an}}$)**.** *Let $V \subseteq \mathbb{C}^n - \{\operatorname{Spec} S\}^{\mathsf{an}}$ be an open set in $\mathbb{C}^n$, then the sections of $\mathcal{O}^{\mathsf{pre}}$ on V is trivial.*

Proof. Let $\sigma \in \Gamma(\mathcal{O}^{\mathsf{pre}}, V)$, we want to show that $\sigma(x) = 0$ for all $x \in V$. By Lemma 3.23, there exists some open neighbourhood $x \in U \subseteq V$[7] such that sections of $\mathcal{O}^{\mathsf{pre}}$ on any generalized polydiscs contained in U is trivial. Thus we can cover V by a family of generalized polydiscs Δ_i such that $\sigma |_{\Delta_i}$ are all zero; therefore by sheaf axioms, σ is zero. □

Corollary 3.25. *Let $V \subseteq \mathbb{C}^n$ be an open subset then we have an isomorphism*

$$\Gamma\left(\mathcal{O}^{\mathsf{pre}}, V \cup (\mathbb{C}^n - \{\operatorname{Spec} S\}^{\mathsf{an}})\right) \cong \Gamma(\mathcal{O}^{\mathsf{pre}}, V)$$

Proof. Write $A := \mathbb{C}^n - \{\operatorname{Spec} S\}^{\mathsf{an}}$, we need to prove the restriction map from $V \cup A$ to V is both injective and surjective.

- Injectivity: suppose a section $\sigma \in \Gamma(\mathcal{O}^{\mathsf{pre}}, V \cup A)$ is in the kernel of the restriction map, in another word, $\sigma |_V$ is zero. Then if we cover $V \cup A$ by V and A, we would know that $\sigma |_A$ is zero as well[8], so by sheaf axiom, σ is zero.
- Surjectivity: let σ be a section in $\Gamma(\mathcal{O}^{\mathsf{pre}}, V)$, then we can glue σ and $0 \in \Gamma(\mathcal{O}^{\mathsf{pre}}, A)$ because $\sigma |_{V \cap A}$ must be zero, since $\Gamma(\mathcal{O}^{\mathsf{pre}}, V \cap A)$ is trivial by Corollary 3.24.

□

Definition 3.26 (Analytification of sheaf)**.** *The analytification $\{\tilde{S}\}^{\mathsf{an}}$ of $\tilde{S}$ is a presheaf on $\{\operatorname{Spec} S\}^{\mathsf{an}}$ defined by the following:*

- *For any open set $U \subseteq \{\operatorname{Spec} S\}^{\mathsf{an}}$, we define the sections $\Gamma\left(\{\tilde{S}\}^{\mathsf{an}}, U\right)$ to be*

$$\Gamma\left(\mathcal{O}^{\mathsf{pre}}, U \cup \left(\mathbb{C}^n - \{\operatorname{Spec} S\}^{\mathsf{an}}\right)\right).$$

- *For any open sets $U \subseteq V \subseteq \{\operatorname{Spec} S\}^{\mathsf{an}}$, we define the restriction map of $\{\tilde{S}\}^{\mathsf{an}}$ from V to U is the restriction map of $\mathcal{O}^{\mathsf{pre}}$ from $V \cup \left(\mathbb{C}^n - \{\operatorname{Spec} S\}^{\mathsf{an}}\right)$ to $U \cup \left(\mathbb{C}^n - \{\operatorname{Spec} S\}^{\mathsf{an}}\right)$.*

This defines a presheaf, the satisfaction of sheaf axioms are essentially from that of $\mathcal{O}^{\mathsf{pre}}$.

Remark 3.27 (Independence of the generators $f_1, \ldots, f_m$)**.** *Though the definition of $\mathcal{O}^{\mathsf{pre}}$ explicity uses f_i's, we see that on the basis of generalized polydiscs, the sections on a generalized polydisc is $\Gamma(\mathcal{H}ol, \Delta) \big/ \ker\theta \cdot \Gamma(\mathcal{H}ol, \Delta)$ which does not mention any generators. Since two sheaves are isomorphic if their sections on a basis are isomorphic, we must conclude that $\mathcal{O}^{\mathsf{pre}}$ is independent of the choice of $f_1, \ldots, f_m$ as well. But we don't know yet if the construction is independent from the choice of $a_1, \ldots, a_m$.*

[7] if U is not a subset of V, then use $U \cap V$

[8] by Corollary 3.24 and that A is open

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